EFFICIENT ADAPTIVE IMPORTANCE SAMPLING FOR TIME-DEPENDENT RELIABILITY ANALYSIS OF STRUCTURES

David Y. Yang[1], J.G. Teng[1,*] and Dan M. Frangopol[2]
[1]Department of Civil and Environmental Engineering,
The Hong Kong Polytechnic University, Hong Kong, China.
[2]Department of Civil and Environmental Engineering, ATLSS Engineering Research Center, Lehigh University,
117 ATLSS Drive, Bethlehem, PA 18015-4729, USA
*Corresponding author; Email: cejgteng@polyu.edu.hk

ABSTRACT

Various methods have been used by researchers to evaluate the time-dependent reliability of structures. Among them, the stochastic-process-based method is theoretically the most rigorous but also computationally the most expensive. To enable the wide application of the stochastic-process-based method in the time-dependent reliability analysis of complex problems, an efficient importance sampling method is presented. This new method, extended from an existing method for time-independent reliability analysis, offers an efficient solution for time-dependent problems of structural systems with multiple important regions. Furthermore, to enhance the efficiency and robustness of the proposed method, a number of numerical measures are proposed. The capability and efficiency of the proposed method are demonstrated through two numerical examples.

KEYWORDS

Time-dependent reliability, adaptive importance sampling, cross entropy, Gaussian mixture, Monte Carlo simulation, system reliability.

INTRODUCTION

Structures experience deterioration and varying load effects during their service life, so the reliability of a structure during its lifetime is a time-independent problem. For this reason, time-dependent reliability analysis has been used by many researchers to evaluate the structural safety of structural systems throughout their service life. Different researchers have adopted different methods in their analysis, and these methods can, in some cases, lead to dramatically different results(Yang et al., 2015). The theoretically rigorous method(i.e. the stochastic-process-based method) for time-dependent reliability analysis usually involves tedious computations, which hinders their application to complex reliability problems with multiple important regions (e.g. time-dependent reliability of structural systems). The objective of this paper is to propose a new and efficient sampling method to facilitate the wide application of the stochastic-process-based method in complex time-dependent reliability problems.

During the service life of a structure, its resistance is likely to deteriorate due to factors such as material degradations (e.g. steel corrosion) and damage from overloading or natural disasters. In addition, the load that a structure has to resist may change over time. Therefore, the performance function of a structure is time-dependent:

$$g(t) = R(t) - S(t) \tag{1}$$

where $R(t)$ and $S(t)$ are the time-dependent resistance and load effect, respectively. In order to assess structural safety during the entire service life, various methods have been proposed by researchers(Melchers, 1999), e.g. the point-in-time reliability method, the time-integrated method, and the stochastic-process-based method. Among these methods ,the stochastic-process-based method is the most accurate and assumes that the arrival of live load events follows a discrete stochastic process(Yang et al., 2015). Using this method, the time-dependent failure probability can be computed using the conditional probability theory as follows:

$$P_f\left(t\mid\mathbf{x}_c\right)=\sum_{k=1}^{\infty}\Pr\left[g\left(\tau;\mathbf{x}_v\left(\tau\right),\mathbf{x}_c\right)<0,\tau<t\mid\mathbf{x}_c,k\right]\Pr\left[N\left(t\right)=k\right]$$
$$P_f\left(t\right)=\int_{\Omega}P_f\left(t\mid\mathbf{x}_c\right)f_{\mathbf{X}_c}\left(\mathbf{x}_c\right)d\mathbf{x}_c \tag{2}$$

where $N(t)=k$ represents the situation that k load events have occurred prior to time t; $\mathbf{x}_v$ and $\mathbf{x}_c$ are the time-dependent and the time-independent variables respectively; $f_{\mathbf{X}c}$ is the probability density function (PDF) of $\mathbf{x}_c$; and τ is any time prior to the time of interest, t. This method was first proposed by Mori and Ellingwood(1993) and has thereafter been extensively used in the time-dependent reliability analysis of deteriorating infrastructure (Enright and Frangopol, 1999; Lounis and Amleh, 2003; Ellingwood, 2005; Akiyama et al., 2010; Okasha and Frangopol, 2010).

Eq. 2means that the performance function should not be very complex. For time-dependent reliability analysis, the performance function can usually be simplified to either Eq. 1 or the following equation:

$$g\left(t\right)=R\left(t\right)-S_D-S_L\left(t\right) \tag{3}$$

where the time-independent dead load effect S_D is isolated from the time-dependent load effect. The integration in Eq. 2usually needs to be carried out with an efficient simulation method such as the adaptive importance sampling method(Mori and Ellingwood, 1993). Although the conventional adaptive importance sampling method(Mori and Ellingwood, 1993; Enright and Frangopol, 1999) can effectively reduce the computational burdenfor simple problems, the stochastic-process-based method still faces computational difficulties, especially for those problems involving multiple random variables and multiple important regions, e.g. in system reliability problems. This is mainly because the method(Mori and Ellingwood, 1993) employs a unimodal sampling function that cannot generate samples efficiently when the actual regions of importance are multimodal. To facilitate the application of the stochastic-process-based method in complex problems, a cross-entropy-based adaptive sampling method using Gaussian mixture is proposed in this paper. The method is an extension of Kurtz and Song's (2013) method for time-independent reliability problems to time-dependent domains. The proposed method also includes a number of improvements formulated to enhance the efficiency and robustness of the original method developed by Kurtz and Song(2013). Two numerical examples are given to illustrate the efficiency of the new method.

CROSS-ENTROPY-BASED ADAPTIVE IMPORTANCE SAMPLING USING GAUSSIAN MIXTURE

Generally, the probability of failure is extremely low in structural reliability problems, resulting in the low efficiency of crude Monte Carlo simulation. In order to improve efficiency, importance sampling has often been used by switching the sampling effort to the more important region(s)with $h_V(\mathbf{x};\mathbf{v})$, the PDF of a new sampling function with parameters $\mathbf{v}$:

$$I=\int_{\Omega}\frac{H\left(\mathbf{x}_c\right)f_{\mathbf{X}}\left(\mathbf{x}_c;\mathbf{u}\right)}{h_V\left(\mathbf{x}_c;\mathbf{v}\right)}h_V\left(\mathbf{x}_c;\mathbf{v}\right)d\mathbf{x}_c \tag{4}$$

where $H(\mathbf{x}_c)=P_f(t|\mathbf{x}_c)$for time-dependent reliability problems. The estimated failure probability and its variance are as follows:

$$\hat{P}_f=\frac{1}{n}\sum_{k=1}^{n}\frac{H\left(\mathbf{x}_k\right)f_{\mathbf{X}}\left(\mathbf{x}_k\right)}{h_V\left(\mathbf{x}_k\right)} \tag{5}$$

$$S_{\hat{P}_f}^2=\frac{1}{n\left(n-1\right)}\sum_{k=1}^{n}\left[\frac{H\left(\mathbf{x}_k\right)f_{\mathbf{X}}\left(\mathbf{x}_k\right)}{h_V\left(\mathbf{x}_k\right)}-\hat{P}_f\right]^2 \tag{6}$$

The number of samples, n, can be several orders smaller than that in crude Monte Carlo simulation if $h_V(\mathbf{x};\mathbf{v})$is reasonably selected. Importance sampling methods have been widely used in complex time-independent reliability problems(Harbitz, 1986; Schuëller and Stix, 1987; Melchers, 1989; Kurtz and Song, 2013). However, these existing importance sampling methods for time-independent reliability problems are inappropriate for time-dependent reliability analysis because the shape or the location of the important region(s) cannot be easily anticipated for time-dependent reliability analysis.

Imposing $S_{\hat{P}_f}^2=0$ in Eq. 6leads to the following expression for the optimal sampling function $h_{Vopt}(\mathbf{x})$:

$$h_{Vopt}\left(\mathbf{x}\right)=\frac{H\left(\mathbf{x}\right)f_{\mathbf{X}}\left(\mathbf{x}\right)}{P_f} \tag{7}$$

which indicates that only one single sample is needed to estimate the probability of failure. Though the direct use of Eq.7 is impossible due to the unknown P_f, a close approximation of the optimal sampling function can significantly enhance sampling efficiency. To achieve this objective, an iterative process can be used to update a sampling kernel progressively to approach a near-optimal sampling function. This category of importance sampling techniques is referred to as adaptive importance sampling. For time-dependent reliability analysis, Mori and Ellingwood (1993) utilized this type of methods to look for the location (i.e. the mean vector $\boldsymbol{\mu}_{Vopt}$)of the optimal sampling function iteratively. Bucher (1988) updated adaptively both the best location and the best shape (i.e. the covariance matrix $\boldsymbol{\Sigma}_{Vopt}$) of the optimal sampling function.

Because of the existence of an optimal sampling function, one can restate the problem as an optimization problem. In this case, the Kullback-Leibler cross-entropy, an indication of the difference between two probability densities defined as follows

$$D\left(h_{V_1}(\mathbf{X}),h_{V_2}(\mathbf{X})\right)=E_1\left(\ln\frac{h_{V_1}(\mathbf{X})}{h_{V_2}(\mathbf{X})}\right)=\int_\Omega \ln h_{V_1}(\mathbf{x})h_{V_1}(\mathbf{x})d\mathbf{x}-\int_\Omega \ln h_{V_1}(\mathbf{x})h_{V_2}(\mathbf{x})d\mathbf{x} \tag{8}$$

can be used to formulate the following optimization problem that represents the adaptation process of adaptive importance sampling:

$$\mathbf{v}=\arg\min_{\mathbf{w}} D\left(h_{Vopt}(\mathbf{X}),h_V(\mathbf{X};\mathbf{w})\right) \tag{9}$$

In Eqs 8 and 9, D(h_{V1}(X),h_{V2}(X)) is the Kullback-Leibler cross-entropy between the PDF $h_{V1}(\mathbf{X})$ and PDF $h_{V2}(\mathbf{X})$; E_1[ln($h_{V1}(\mathbf{X})/h_{V2}(\mathbf{X})$)]is the expected value of ln($h_{V1}(\mathbf{X})/h_{V2}(\mathbf{X})$) with **X** being drawn following the PDF $h_{V1}(\mathbf{X})$; and **w** is the group of parameters of sampling function $h_V(\mathbf{X};\mathbf{w})$.

The optimization problem has analytical solutions for distributions in the exponential family(Rubinstein and Kroese, 2004; Kurtz and Song, 2013).Kurtz and Song (2013)applied Eq. 9 to time-independent reliability problems with multiple important regions by using Gaussian mixture given below as the importance sampling kernel:

$$h_V(\mathbf{x};\mathbf{w})=\sum_{j=1}^{n_w}\pi_j N\left(\mathbf{x};\boldsymbol{\mu}_j,\boldsymbol{\Sigma}_j\right) \tag{10}$$

where n_w is the total number of Gaussian components; $N(\mathbf{x};\boldsymbol{\mu}_j;\boldsymbol{\Sigma}_j)$is the PDF of the j-th multivariate Gaussian distribution with a mean vector $\boldsymbol{\mu}_j$ and a covariance matrix $\boldsymbol{\Sigma}_j$;π_j, with $\sum_{j=1}^{n_w}\pi_j=1$and $0\le \pi_j \le 1$, are weighting factors of the Gaussian components; the parameters **w**, therefore, have($3\times n_w$)components, i.e. $\mathbf{w}=\{\pi_1,\ldots,\pi_{nw},\ldots,\boldsymbol{\mu}_j,\ldots,\boldsymbol{\mu}_{nw},\boldsymbol{\Sigma}_j,\ldots,\boldsymbol{\Sigma}_{nw}\}$. Since a multivariate Gaussian distribution is in the family of exponential distributions, the analytical solution for the optimization problem given by Eq. 9 can be deduced following the same procedures given in the existing studies(Rubinstein and Kroese, 2004; Kurtz and Song, 2013). Eqs 11to 13 show the updating rules for $\boldsymbol{\mu}_j$, $\boldsymbol{\Sigma}_j$ and π_j, j=1,…,n_w(Kurtz and Song, 2013):

$$\boldsymbol{\mu}_j=\frac{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})\gamma_{i,j}\mathbf{x}_i}{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})\gamma_{i,j}} \tag{11}$$

$$\boldsymbol{\Sigma}_j=\frac{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})\gamma_{i,j}\left(\mathbf{x}_i-\boldsymbol{\mu}_j\right)\left(\mathbf{x}_i-\boldsymbol{\mu}_j\right)^T}{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})\gamma_{i,j}} \tag{12}$$

$$\pi_j=\frac{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})\gamma_{i,j}}{\sum_{i=1}^{n_{sub}}H(\mathbf{x}_i)W(\mathbf{x}_i;\mathbf{u},\mathbf{w})} \tag{13}$$

where n_{sub} is the number of simulations in each adaptation step;$W(\mathbf{x};\mathbf{u},\mathbf{w})$is referred to as the likelihood ratio between $f_{\mathbf{X}}$ and h_V, i.e.

$$W(\mathbf{x};\mathbf{u},\mathbf{w})=\frac{f_{\mathbf{X}}(\mathbf{x};\mathbf{u})}{h_V(\mathbf{x};\mathbf{w})} \tag{14}$$

and $\gamma_{i,j}$is referred to as the "responsibility" of the component distribution $N(\mathbf{x};\boldsymbol{\mu}_j,\boldsymbol{\Sigma}_j)$with respect to the observation $\mathbf{x}_i$, as defined in the following equation:

$$\gamma_{i,j}=\Pr\left[\mathbf{X}\sim N\left(\mathbf{X};\boldsymbol{\mu}_j,\boldsymbol{\Sigma}_j\right)\mid \mathbf{X}=\mathbf{x}_i\right]=\frac{\pi_j N\left(\mathbf{x}_i;\boldsymbol{\mu}_j,\boldsymbol{\Sigma}_j\right)}{\sum_{k=1}^{n_w}\pi_k N\left(\mathbf{x}_i;\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k\right)} \tag{15}$$

The detailed deduction of Eqs 11 to 15 can be found in the aforementioned studies(Rubinstein and Kroese, 2004; Kurtz and Song, 2013).Eqs11 to 15 provide a clear and explicit adaptation procedure to approach the near-optimal sampling function. Following Eqs11 to 15, parameters $\mathbf{w}$ will adapt to the optimal parameters $\mathbf{v}$ in Eq. 9, leading to an efficient sampling function that can dramatically reduce the computational cost. In particular, in each adaptation step (i.e. step i), $\mathrm{w}^{(i)} = \{\pi_1^{(i)}, \ldots, \pi_{nw}^{(i)}, \boldsymbol{\mu}_1^{(i)}, \ldots, \boldsymbol{\mu}_{nw}^{(i)}, \boldsymbol{\Sigma}_1^{(i)}, \ldots, \boldsymbol{\Sigma}_{nw}^{(i)}\}$can be determined using Eqs11 to 15 with$\mathrm{w} = w^{(i-1)}$.Usually, the adaptation process converges to the optimal values after only a few steps.

Additionally, it should be noted that the selection of the initial $\mathbf{w}$ plays an important role in the adaptation process, and possible singularity issues in Eqs11 to 15 can create some difficulties in the numerical implementation. Such computational difficulties have also been noticed by Kurtz and Song(2013), but they did not provide a plausible solution. The numerical robustness issue is further discussed and tackled later in the paper. Finally, it should be mentioned that if a unimodal sampling function is used, i.e.$\gamma_{i,j}$=1andn_w=1,thecross-entropy-based method (i.e. Eqs 11 to 15) reduces to Mori and Ellingwood's(1993)or Bucher's (1988)method, depending on whether only Eq. 11or both Eq. 11 and Eq. 12 are used in the adaptation process. As can be seen later in the numerical examples, the present method is more advantageous than Mori and Ellingwood'sor Bucher's method because it is able to accommodate multiple important regions efficiently in system reliability problems.

TIME-DEPENDENT RELIABILITY ANALYSIS WITH CROSS-ENTROPY-BASED SAMPLING METHOD

Time-Dependent Reliability of Structures and Structural Systems

As stated earlier, the live load should be considered as a stochastic process. In this paper, the time-dependent reliability of bridges is considered as bridges may experience severe deteriorations over their service life. The live load process of a bridge can be regarded as a Poisson process with arrival rate λ. The structural resistance is likely to decrease due to environmental attacks. If structural deterioration is represented by a deterioration function $g_R(t)$, the performance function of a time-dependent reliability problem can be re-written as

$$g(t) = R_0 g_R(t) - S_D - S_L(t) \tag{16}$$

whereR_0 is the initial resistance (time-independent); $g_R(t)$ is the deterministic deterioration function; S_D is the dead load effect (time-independent); and $S_L(t)$is the live load effect (time-dependent).The conditional time-dependent failure probability of RC bridge girders can be expressed as (Mori and Ellingwood, 1993; Enright and Frangopol, 1999):

$$P_f(t \mid r, s_D) = 1 - \exp\left\{-\lambda t\left[1 - \frac{1}{t}\int_0^t F_{S_L}\left(r g_R(\tau) - s_D\right) d\tau\right]\right\} \tag{17}$$

where r and s_D are observations of R_0 and S_D; and F_{SL} is the cumulative distribution function (CDF) at each live load arrival. Eq. 17 can be further simplified forseries systems. According to Mori and Ellingwood (1993), the conditional failure probability of a series structural system can be calculated as

$$P_f(t \mid \mathbf{r}, \mathbf{s}_D) \approx P_f(t \mid \mathbf{r}) = 1 - \exp\left\{-\lambda t\left[1 - \frac{1}{t}\int_0^t F_S\left(\min_{i=1}^{m} \frac{r_i g_{R_i}(\tau)}{c_i}\right) d\tau\right]\right\} \tag{18}$$

where $c_i$arethe distribution factors of live load [e.g. girder distribution factors (GDFs) in bridge superstructures];and m is the number of elements in the series system. It should be noted thatthe uncertainty of dead load has little influence on the time-dependent reliability of bridge superstructures (Mori and Ellingwood, 1993; Enright and Frangopol, 1999). Therefore, the CDFs of $S_L$havebeen substituted by those of the total load effect S in Eq. 18.In the time-dependent reliability analysis, Eqs17 and18 are evaluated by numerical integration, while the total probability of failure is computed with a simulation method. For system reliability problems, multiple important regionsmay occur(Mori and Ellingwood, 1993), which will compromise the efficiency of existingmethods. The above cross-entropy-based sampling method using Gaussian mixture provides anefficientalternative in such situations.

Numerical Robustness of Cross-Entropy-Based Adaptive Importance Sampling Using Gaussian Mixture

Initial Parameters of Gaussian Mixture

The initial parameters of Gaussian mixture, $\mathbf{w}$, play an important role in the efficiency and robustness of the new method. A good guess of important regions can dramatically increase the speed of convergence and decrease the required number of samples while a poor guess can either increase the computational cost or cause convergence to only some of the important regions instead of all of them. Kurtz and Song's(2013) method has two major deficiencies in the selection of the initial $\mathbf{w}$ (Yang et al., 2015).To correct these deficiencies, it is recommended herein that the Latin hypercube sampling method be used to generate $\boldsymbol{\mu}_j$ in Eq. 10. As an efficient stratified sampling method, Latin hypercube sampling is capable of generating representative values of the whole sample space. Correlation between random variables is controlled with simulated annealing (Vořechovský, 2004). The target correlation matrix can be defined according to the configuration of the considered structural system. For a series system, a negative-correlation matrix can be used as the target correlation matrix. Higham's(2002) algorithm is employed to find the nearest correlation matrix if the proposed correlation matrix is not positive definitive. This algorithm is also able to deal with systems with elements of different degrees of importance.

Covariance Updating and Control

During the adaptation process, the covariance of samples may shrink or expand to mimic the optimal sampling function. This process does give a good approximation of the optimal sampling function. However, as indicated by Melchers(1990) and shown in numerical example 1 given later, the estimated failure probability may oscillate around the real value when the covariance of the sampling function becomes too small. It is also inefficient to use a larger standard deviation (STD) to overcome the problem as this increases the chance of obtaining sample points of low importance. Moreover, for Gaussian mixture adaptations, a larger STD of sampling function may result in the clustering of Gaussian mixture components.

In the new method proposed in this paper, n_k samples will be generated after the adaptation process in order to determine k_{opt} which is defined as follows:

$$k_{opt} = k_{opt,ij} = \sqrt{\frac{\Sigma_{kij}}{\Sigma_{pre,kij}}} \quad \text{where } i, j = 1, \ldots, n_{RV} \tag{19}$$

where Σ_{kij}areelements of the covariance matrix of the k-th Gaussian component, and $\Sigma_{pre,kij}$ are the elements of the covariance matrix of the k-th Gaussian component right after the preliminary sampling (i.e. the sampling to arrive at the near-optimal sampling function).Based on the results of a trial-and-error process, it is recommended herein that k_{opt} be reduced from 2.00 to 1.00at an interval of 0.05 as long as the simulated failure probability from n_k samples does not drop dramatically(Yang et al., 2015).This control process can mitigate the observed oscillation while preserving the computational efficiency.

Besides the preceding covariance control, the covariance matrix should also be expanded during the first few steps of adaptation in the preliminary sampling so that there are enough points falling into the important region(s). For unimodal sampling functions, Mori and Ellingwood (1993) recommended that $\sigma_{V,i}^{(s)} = k_s \sigma_{Xi}$, where $\mathrm{i} = 1, \ldots, \mathrm{n}_{RV}$ with

$$k_s = \begin{cases} 3.0 & \text{where } s = 1 \text{ for the first adaptation step} \\ 2.2 & \text{where } s = 2,3 \text{ for the second and third adaptation steps} \\ 1.6 & \text{where } s \geq 4 \text{ for the other adaptation steps until convergence} \end{cases} \tag{20}$$

According to Yang et al.(2015), if Gaussian mixture is used, it is proposed that$\Sigma_{kij}^{(s)} = k_s^2 \, \Sigma_{pre,kij}$, where $\mathrm{i} = 1, \ldots, \mathrm{n}_{RV}$with

$$k_s = \begin{cases} 3.0 & \text{where } s = 1 \text{ for the first adaptation step} \\ 1.5 & \text{where } s = 2,3 \text{ for the second and third adaptation steps} \\ 1.2 & \text{where } s = 4,5 \text{ for the fourth and fifth adaptation steps} \\ 1.0 & \text{where } s \geq 6 \text{ for the other adaptation steps until convergence} \end{cases} \tag{21}$$

According to the preceding discussion, the cross-entropy-based adaptive importance sampling method needs two cycles of sampling: preliminary sampling and main sampling. During the preliminary sampling cycle, adaptation is conducted to obtain the near-optimal sampling function. During the main sampling cycle, only a relatively small number of samples are needed to predict failure probability. The results from both cycles of sampling are then combined to obtain an unbiased estimation of failure probability as follows(Mori and Ellingwood, 1993):

$$\hat{w} = \frac{S^2_{\hat{P}_f, main}}{S^2_{\hat{P}_f, pre} + S^2_{\hat{P}_f, main}} \tag{22}$$

$$\hat{P}_f = \left(1 - \hat{w}\right) \hat{P}_{f,main} + \hat{w} \hat{P}_{f,pre} \tag{23}$$

$$S^2_{\hat{P}_f} = \left(1 - \hat{w}\right)^2 S^2_{\hat{P}_{f,main}} + \hat{w}^2 S^2_{\hat{P}_{f,pre}} \tag{24}$$

where $\hat{P}_{f,pre}$ and $S^2_{\hat{P}_{f,pre}}$ as well as $\hat{P}_{f,main}$ and $S^2_{\hat{P}_{f,main}}$ are the estimated probabilities of failure and their corresponding variances from the preliminary and the main sampling cycles, respectively.

NUMERICAL EXAMPLES

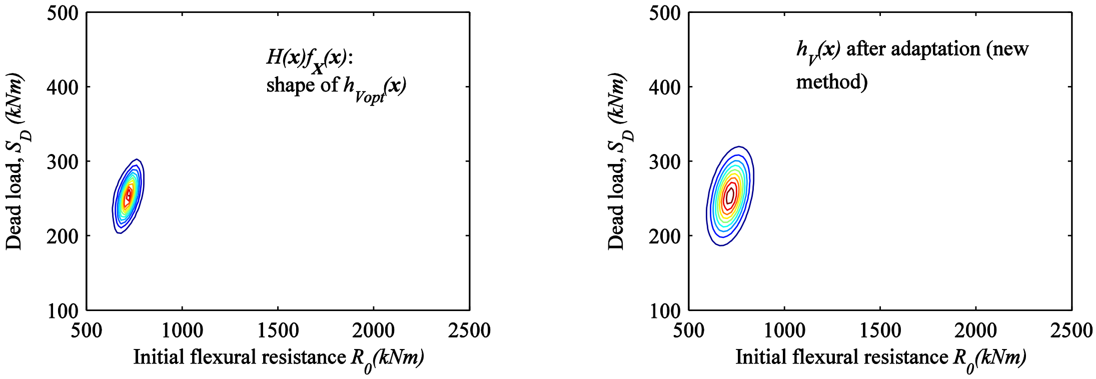
Example 1: Importance of Covariance Adaptation and Control

Mori and Ellingwood's (1993)method has been widely used in time-dependent reliability analysis. However, this method only updates the location of the sampling function during each adaptation step. This can jeopardize the efficiency of the method when the optimal importance sampling function is highly skewed. In addition, as stated earlier, covariance control is important when covariance adaptation is implemented. Through this example, the importance of covariance adaptation and control is demonstrated. The example is a deteriorating RC bridge girder. The initial resistance R_0 follows a lognormal distribution with a mean equal to 1573 kNm and a COV of 0.17; the dead load S_D follows a normal distribution with a mean of 233.6 kNm and a COV of 0.10; and the live load S_L at each arrival is a normal random variable with a mean of 293.3 kNm and a COV of 0.40. The arrival rate of the live load is 1000 times/year, i.e.λ=1000. Three sampling algorithms were implemented, i.e. Mori and Ellingwood's method, Bucher's method (i.e. covariance adaptation without control), and the new method (covariance adaptation with control). Reliability at the end of the first year was evaluated using all three methods. Herein, only one year is considered so that a skewed import region can be obtained for comparison purpose. In practice, the proposed method can be used for time-dependent reliability analysis of much longer time. During the first year, structural deterioration does not occur, so $g_R(t)$=1. With each method, 12 runs of analysis were undertaken in order to provide a more reliable assessment of the methods. All the results are listed in Table 1.

Table 1Comparison of different methods for problems with skewed important regions

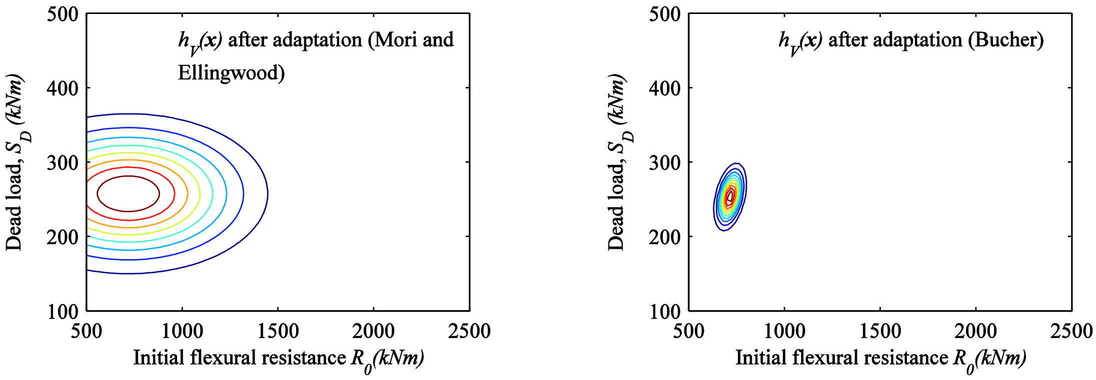
Run #	Bucher (1988)		Mori and Ellingwood (1993)		New method	
	P_f	δ_{Pf}(%)	P_f	δ_{Pf} (%)	P_f	δ_{Pf} (%)
1	1.91e-4	1.88	2.10e-4	9.00	2.05e-4	1.60
2	1.79e-4	4.73	2.14e-4	9.19	2.08e-4	1.60
3	2.04e-4	0.74	2.17e-4	8.89	2.02e-4	0.73
4	4.05e-5	18.9	1.98e-4	8.28	2.03e-4	0.82
5	2.01e-4	1.04	1.95e-4	9.45	2.00e-4	0.99
6	2.06e-4	1.12	1.61e-4	8.72	2.05e-4	1.14
7	2.14e-4	3.14	2.15e-4	8.77	2.05e-4	1.44
8	2.02e-4	0.93	2.18e-4	9.56	2.08-e4	0.97
9	2.02e-4	1.10	1.86e-4	8.37	2.02e-4	0.92
10	7.79e-5	15.62	2.05e-4	8.61	2.03e-4	0.73
11	2.06e-4	1.90	2.16e-4	8.23	2.09e-4	1.02
12	1.66e-4	1.76	1.88e-4	9.34	2.03e-4	0.76
mean	1.97e-4	1.83	2.02e-4	8.87	2.04e-4	1.06
COV(%)	7.35	-	8.53	-	1.36	-

Though the optimal sampling function cannot be determined a priori, its shape can be illustrated by function $H(\mathbf{x})f_{\mathbf{X}}(\mathbf{x})$, as shown in Figure 1(a). As can be seen, the optimal sampling function is indeed skewed. Adaptation of the sampling function with the new method is presented in Figure 1(b) for one specific run in Table 2. For comparison, the sampling PDFs after adaptation are shown as well for Mori and Ellingwood's and Bucher's methods. From Table 1 and Figures1and2, it can be concluded that covariance adaptation can improve the robustness and efficiency of the sampling method.



(a)Shape of the optimal sampling function (b)Near-optimal sampling function after adaptation

Figure 1 Shape of optimal sampling function and its approximation using the proposed method



(a) Mori and Ellingwood's method (b) Bucher's method

Figure 2 Approximation of the optimal sampling function using the existing methods

Example 2: Sampling Using Gaussian Mixture and Time-Dependent Reliability of Series Structural Systems

Gaussian mixture is capable of accommodating multiple peaks of the optimal sampling function and is thus more efficient in system reliability problems. In this example, the time-dependent reliability analysis of a series structural system was conducted to demonstrate the high efficiency of the new method. The system is the superstructure of an RC girder bridge that consists of five girders (Enright and Frangopol, 1999). The system is modelled as a series system, i.e. the failure of any girder indicates the failure of the whole superstructure. Eq. 18 is used in this example. The initial resistance R_0 follows a lognormal distribution with a mean equal to 1790 kNm and a COV of 0.16; the dead load is a deterministic variable equal to 231.2 kNm; and the live load S_L at each arrival is a normal random variable with a mean of 301.4 kNm and a COV of 0.40. The arrival rate of live load is 1000 times/year. In the analysis, it was assumed that all the bridge girders have identical flexural resistance.

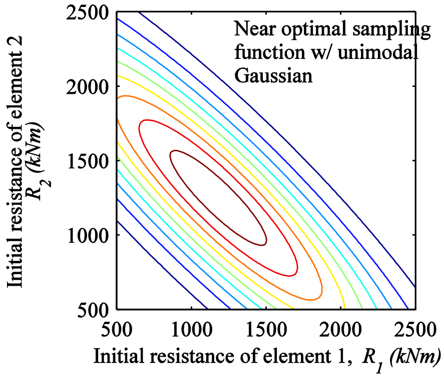
The time-dependent reliability of four series structural systems was analyzed using Mori and Ellingwood's(1993)method, the cross-entropy-based method with a unimodal sampling function, and the cross-entropy-based method with Gaussian mixture. These series systems are composed of 2 to 5 elements, and c_i in Eq. 18 is assumed to be 0.51 for all elements in the series systems. Reliability at the end of the first year was evaluated. Similar to the first example, the one year period used in this example is to ensure that the important regions are positioned separately from each other. Since deterioration is not likely to occur in the first year, $g_R(t)$=1forall four systems. The efficiency that can be achieved using Gaussian mixture was examined. Table 2 gives the analysis results. It can be observed that cross-entropy-based importance sampling using Gaussian mixture can significantly increase the sampling efficiency.

Table 2Comparison of computation efficiency for series systems

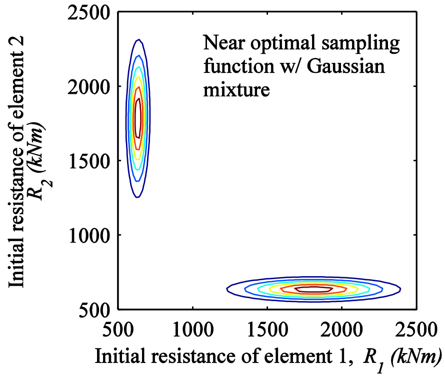
Element	Method[1]	Sampling parameters[2]					Main sampling		Final results	
		n_w	n_{sub}	n_{adp}	n_{main}	n_{total}	$P_{f,main}$	$\delta_{pf,main}$	P_f	δ_{pf}
2	M&E	-	1000	20	2000	22000	1.39e-11	0.137	1.25e-11	0.042
	Uni	-	1000	20	2000	22000	1.34e-11	0.067	1.26e-11	0.027
	Mixture	4	400	10	200	4200	1.23e-11	0.016	1.23e-11	0.015
3	M&E	-	1000	20	3000	23000	1.49e-11	0.165	1.79e-11	0.074
	Uni	-	1000	20	3000	23000	2.02e-11	0.092	1.84e-11	0.040
	Mixture	6	600	11	300	6900	1.89e-11	0.028	1.85e-11	0.027
4	M&E	-	1500	20	4000	34000	2.23e-11	0.146	2.38e-11	0.055
	Uni	-	1500	20	4000	34000	2.06e-11	0.111	2.17e-11	0.048
	Mixture	8	800	11	400	9200	2.47e-11	0.032	2.45e-11	0.030
5	M&E	-	3000	20	10000	70000	2.74e-11	0.094	2.87e-11	0.034
	Uni	-	3000	20	10000	70000	3.35e-11	0.093	3.00e-11	0.043
	Mixture	25	2500	13	500	33000	2.92e-11	0.019	2.93e-11	0.018

Note:
[1] "M&E" = Mori and Ellingwood's (1993) method; "Uni" = cross-entropy-based method with a unimodal Gaussian distribution; and "Mixture" = cross-entropy-based method with Gaussian mixture.
[2] n_w= number of Gaussian components; n_{adp}= number of adaptation steps; n_{sub}= number of samples in each step of adaptation; n_{main}= number of samples during main sampling; n_{total}= total number of samples.

Figure 3 shows the near-optimal sampling function after adaptation for the series system with two elements. It can be seen that Gaussian mixture is able to cover all the important regions efficiently, which can explain the smaller n_{main} and the lower $\delta_{Pf,main}$ in Table 2.



(a)Unimodal Gaussian



(b) Gaussian mixture

Figure 3 Near-optimal sampling functions after adaptation (2 elements)

CONCLUSIONS

A cross-entropy-based adaptive importance sampling method has been proposed for the efficient computation of time-dependent reliability of structural systems. The method uses Gaussian mixture to accommodate multiple important regions that may occur in structural systems. From the results and discussions presented in the paper, the following conclusions can be drawn:

1. The proposed sampling method is more efficient than the existing methods, especially for series systems with multiple important regions.
2. With Gaussian mixture as the sampling kernel, a multimodal near-optimal sampling function can be obtained after only a few steps of adaptation.
3. A number of numerical measures were proposed and shown to improve the efficiency and robustness of the proposed sampling method; these include the use of Latin hypercube sampling, simulated annealing, appropriate design of the target correlation matrix, and updating of this matrix using Higham's algorithm.
4. It is important to control correlation adaptation during preliminary sampling in order to eliminate possible oscillations of the estimated failure probability.

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